L1 - Quantum Mechanics of Lattice Discretized Scalar Field Theory

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STANDARD TEXTBOOKS

Classical Field Theory with action $S(\varphi)$

$\xrightarrow{\text{Legendre transform}}$ Hamiltonian formalism for Classical Field Theory

$\downarrow$ Quantization by path integral

Quantum Field Theory

$$\langle A \rangle = \frac{1}{Z} \int \mathcal{D}\varphi \, e^{iS(\varphi)} A(\varphi)$$

$\downarrow$ Wick rotation

Euclidean QFT

$$\langle A \rangle = \frac{1}{Z} \int \mathcal{D}\varphi \, e^{-S_E(\varphi)} A(\varphi)$$

$\downarrow$ Discretization on 4d lattice

Lattice QFT

THIS COURSE

Hamiltonian formalism for Classical Field Theory

$\downarrow$ Canonical quantization

Hamiltonian formalism for QFT

$\downarrow$ Discretization on 3d lattice

Hamiltonian formalism for QFT on a 3D lattice

$\downarrow$ Wick rotation

Hamiltonian formalism for Euclidean QFT on a 3D lattice

$\xrightarrow{\text{Time-disretized path integral}}$ Lattice QFT

Classical φ^4 theory - Action formalism

φ real scalar field $\qquad \varphi : \mathbb{R}^4 \mapsto \mathbb{R}$

$\qquad\qquad\qquad\qquad\qquad x \mapsto \varphi(x)$

$$\mathcal{L}\left(\varphi(x), \partial\varphi(x), \text{higher derivatives in } x\right) = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi(x) - \frac{m^2}{2}\varphi^2(x) - \frac{\lambda}{4!}\varphi^4(x)$$

Lagrangian density $\qquad g = \mathrm{diag}(+1,-1,-1,-1)$

$$S_V(\varphi) = \int_V d^4x\ \mathcal{L}(\varphi(x), \partial\varphi(x), \dots)$$

Action on domain V

φ satisfies the equations of motion $\iff$ $S_V(\varphi+\eta) = S_V(\varphi) + O(\eta)^2$ for every η with compact support in the interior of V $\iff$ $\dfrac{\delta S_V}{\delta\varphi(x)} = 0$ for every $x \in \mathrm{int}(V)$

$$S_V(\varphi+\eta) = \dots = S_V(\varphi) + \int_V d^4x\ \left[\partial_\mu\varphi\,\partial^\mu\eta - m^2\varphi\eta - \frac{\lambda}{3!}\varphi^3\eta\right](x) \quad + O(\eta)^2$$

$$= S_V(\varphi) + \underbrace{\oint_{\partial V} dS^\mu\,\eta\,\partial_\mu\varphi}_{\substack{\eta=0 \text{ on the boundary}\\ \text{of V by hypothesis}}} - \underbrace{\int_V d^4x\ \eta(x)\left[\partial_\mu\partial^\mu\varphi + m^2\varphi + \frac{\lambda}{3!}\varphi^3\right](x)}_{\text{this vanishes for every } \eta} + O(\eta)^2$$

$\iff \partial_\mu\partial^\mu\varphi + m^2\varphi + \frac{\lambda}{3!}\varphi^3 = 0 \qquad$ (equations of motion)

Classical φ^4 theory – Hamiltonian formalism

Time and space are treated differently $\qquad \varphi_t(\underline{x}) = \varphi(t,\underline{x}) \qquad \dot{\varphi}_t(\underline{x}) = \frac{\partial}{\partial t}\varphi(t,\underline{x})$

$$L(\varphi_t, \dot{\varphi}_t) = \int d^3x \; \mathcal{L}(\varphi(t,\underline{x}), \partial\varphi(t,x))$$

Lagrangian

$$= \int d^3x \left[\frac{1}{2}\dot{\varphi}_t^2 - \frac{1}{2}\sum_{k=1}^{3}\partial_k\varphi_t\,\partial_k\varphi_t - \frac{m^2}{2}\varphi_t^2 - \frac{\lambda}{4!}\varphi_t^4\right](\underline{x}) = \int d^3x \; \frac{1}{2}\dot{\varphi}_t^2 \; - \; V(\varphi_t)$$

Canonical momentum associated to φ_t: $\qquad \pi_t(\underline{x}) = \frac{\delta L}{\delta\dot{\varphi}_t(\underline{x})} = \dot{\varphi}_t(\underline{x})$

Hamiltonian = Legendre transform of Lagrangian

$$H(\pi_t,\varphi_t) = \int d^3x \; \pi_t(\underline{x})\dot{\varphi}_t(\underline{x}) - L(\varphi_t,\dot{\varphi}_t) = \int d^3x \; \frac{1}{2}\pi_t^2(\underline{x}) + V(\varphi_t)$$

$$= \int d^3x \left[\frac{1}{2}\pi_t^2 + \frac{1}{2}(\underline{\nabla}\varphi_t)^2 + \frac{m^2}{2}\varphi_t^2 + \frac{\lambda}{4!}\varphi_t^4\right](\underline{x})$$

Equations of motion
$\equiv$ Hamilton equations

$$\begin{cases} \dot{\varphi}_t = \frac{\delta H}{\delta \pi_t} = \pi_t \\ \dot{\pi}_t = -\frac{\delta H}{\delta\varphi_t} = \underline{\nabla}^2\varphi_t - m^2\varphi_t - \frac{\lambda}{3!}\varphi_t^3 \end{cases}$$

Canonical quantization

1. Upgrade $\varphi(\underline{x})$ and $\pi(\underline{x})$ to "operators" on some Hilbert space, satisfying the self-adjoint condition (real fields) and the canonical commutation relations

$$\varphi(\underline{x})^\dagger = \varphi(\underline{x}) \qquad [\varphi(\underline{x}), \pi(\underline{y})] = i\delta^3(\underline{x}-\underline{y})$$
$$\pi(\underline{x})^\dagger = \pi(\underline{x}) \qquad [\varphi(\underline{x}), \varphi(\underline{y})] = [\pi(\underline{x}), \pi(\underline{y})] = 0$$

2. The Hamiltonian is formally identical to the classical one:

$$H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\underline{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4\right](\underline{x})$$

3. The time evolution follows the standard rule of Quantum Mechanics.

- Schrödinger picture — states evolve $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$; observables don't $A \quad \varphi(\underline{x}) \quad \pi(\underline{x}) \ \ldots$

- Heisenberg picture — observables evolve $A(t) = e^{iHt} A e^{-iHt}$

$$\varphi(t,\underline{x}) = e^{iHt}\varphi(\underline{x})e^{-iHt} \qquad \pi(t,\underline{x}) = e^{iHt}\pi(\underline{x})e^{-iHt}$$

  states don't $|\psi\rangle$

- Interaction picture (useful for perturbative expansion, we will not use it)

4. What about the Hilbert space?

(a) For the free theory, this is the Fock space

(b) In the general interacting case, the explicit construction of the Hilbert space is still an open problem. Physicist usually assume that this Hilbert space exists, and derive properties, general relations... (axiomatic QFT)

$$[\varphi(\underline{x}), \pi(\underline{y})] = i\, \underbrace{\delta^3(\underline{x}-\underline{y}) \times \text{Identity operator}}$$

this is not an operator because $\delta^3(\underline{x}-\underline{y})$ is not a number for any generic $\underline{x}, \underline{y}$: $\delta^3(0) = \infty$!

Two options to deal with this complication:

(a) $\varphi(\underline{x})$ and $\pi(\underline{x})$ are distribution-valued operators

(b) Regularize the theory. The regularization procedure turns $\varphi(\underline{x})$ and $\pi(\underline{x})$ into honest operators. We will follow this approach.

Lattice regularization of (3d) space

Continuous space $\longrightarrow$ Infinite lattice $\longrightarrow$ Finite lattice

$\underline{x} \in \mathbb{R}^3$

$\underline{x} \in a\mathbb{Z}^3$

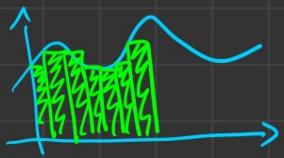
$\underline{x} = (an_1, an_2, an_3) \qquad n_k \in \mathbb{Z}$

$a =$ lattice spacing

$\underline{x} \in a I_N^3 \qquad I_N = \{0, 1, \dots, N-1\}$

$\underline{x} = (an_1, an_2, an_3) \qquad n_k \in I_N$

$|a I_N^3| = \#$ of points $= N^3$



$$\int d^3x \; f(\underline{x}) \longrightarrow \sum_{\underline{x} \in a\mathbb{Z}^3} a^3 f(\underline{x}) \longrightarrow \sum_{\underline{x} \in a I_N^3} a^3 f(\underline{x})$$

$$\delta^3(\underline{x} - \underline{y}) \longrightarrow a^{-3}\, \delta_{\underline{x},\underline{y}} \longrightarrow \text{same}$$

N.B. $\int d^3x \; \delta^3(\underline{x} - \underline{y}) = 1$ $\qquad \sum_{\underline{x}} a^3 \, a^{-3} \delta_{\underline{x},\underline{y}} = 1$

$$\frac{\partial f}{\partial x_k}(\underline{x}) \longrightarrow \partial_k^f f(\underline{x}) = \frac{f(\underline{x} + a\underline{e}_k) - f(\underline{x})}{a} \quad \text{(forward derivative)}$$

$$\partial_k^b f(\underline{x}) = \frac{f(\underline{x}) - f(\underline{x} - a\underline{e}_k)}{a} \quad \text{(backward derivative)}$$

$\longrightarrow$ we choose periodic boundary conditions, i.e. the $\pm$ in $\underline{x} \pm a\underline{e}_k$ must be interpreted modulo aN.

Scalar field theory on N^3 lattice $\equiv$ Q.M. point particle in N^3-dimensional space

Commutation relations

$$[\hat{\varphi}(\underline{x}), \hat{\pi}(\underline{y})] = i\, a^{-3} \delta_{\underline{x},\underline{y}}$$
$$[\hat{\varphi}(\underline{x}), \hat{\varphi}(\underline{y})] = [\hat{\pi}(\underline{x}), \hat{\pi}(\underline{y})] = 0$$

Hamiltonian

$$H = \sum_{\underline{x} \in a I_N^3} a^3 \left[\frac{1}{2} \hat{\pi}^2 + \frac{1}{2} \sum_{k=1}^{3} (\partial_k^f \hat{\varphi})^2 + \frac{m^2}{2} \hat{\varphi}^2 + \frac{\lambda}{4!} \hat{\varphi}^4 \right](\underline{x})$$

Hilbert space

$\mathcal{C}$ = space of field configurations
$= \{ \varphi \text{ s.t. } \varphi : a I_N^3 \to \mathbb{R} \} \simeq \mathbb{R}^{N^3}$

$\mathcal{H}$ = space of normalizable wave functions on $\mathcal{C}$
$= \left\{ \Psi \text{ s.t. } \Psi : \mathcal{C} \to \mathbb{C} \text{ and } \int \left[\prod_{\underline{x}} d\varphi(\underline{x}) \right] |\Psi(\varphi)|^2 < +\infty \right\} \simeq L^2(\mathbb{R}^{N^3})$

- Label the lattice points $\underline{x}$ with an index $\alpha = 0, 1, \dots, N^3 - 1$
- Set $\hat{q}_\alpha = \hat{\varphi}(\underline{x})$, $\hat{p}_\alpha = a^3 \hat{\pi}(\underline{x})$
- $[\hat{q}_\alpha, \hat{p}_\beta] = i \delta_{\alpha\beta}$ $[\hat{q}_\alpha, \hat{q}_\beta] = [\hat{p}_\alpha, \hat{p}_\beta] = 0$

position and momentum operators for the point particle in N^3-dim. space

- $H = \sum_\alpha \frac{\hat{p}_\alpha^2}{2a^3} + V(\hat{q})$
- $L^2(\mathbb{R}^D)$ = space of normalizable wave functions $\Psi(q)$ $(D = N^3)$
$= \{ \Psi \text{ s.t. } \Psi : \mathbb{R}^D \to \mathbb{C} \text{ and } \int d^D q\, |\Psi(q)|^2 < +\infty \}$

Dirac bra-ket notation

$\to$ field operator

$\to$ simoultaneous eigenvector of $\hat{\varphi}(\underline{x})$ for every $\underline{x}$

$\to$ eigenvalue : for each $\underline{x}$, $\varphi(\underline{x})$ is a number

- $\hat{\varphi}(\underline{x})\,|\varphi\rangle = \varphi(\underline{x})\,|\varphi\rangle$ $\qquad$ $\hat{q}_a|q\rangle = q_a|q\rangle$

  $\langle\varphi'|\varphi\rangle = \prod_{\underline{x}} \delta(\varphi'(\underline{x})-\varphi(\underline{x}))$ orthogonality rel. $\qquad$ $\langle q'|q\rangle = \delta^D(q'-q) = \prod_a \delta(q'_a - q_a)$

  $\int \left[\prod_{\underline{x}} d\varphi(\underline{x})\right] |\varphi\rangle\langle\varphi| = I$ completeness rel. $\qquad$ $\int d^Dq\, |q\rangle\langle q| = I$

  $\Psi(\varphi) = \langle\varphi|\Psi\rangle$ wave function associated to $|\Psi\rangle$ $\qquad$ $\psi(q) = \langle q|\psi\rangle$

- $\hat{\pi}(\underline{x})\,|\pi\rangle = \pi(\underline{x})\,|\pi\rangle$ $\qquad$ $\hat{p}_a|p\rangle = p_a|p\rangle$

  $\langle\pi'|\pi\rangle = \prod_{\underline{x}} 2\pi\,\delta(\pi'(\underline{x})-\pi(\underline{x}))$ $\qquad$ $\langle p'|p\rangle = (2\pi)^D\delta^D(p'-p)$

  $\int \left[\prod_{\underline{x}} \frac{d\pi(\underline{x})}{2\pi}\right] |\pi\rangle\langle\pi| = I$ $\qquad$ $\int \frac{d^Dp}{(2\pi)^D} |p\rangle\langle p| = I$

  $\langle\varphi|\pi\rangle = \exp\left\{ i \sum_{\underline{x}} a^3\, \pi(\underline{x})\varphi(\underline{x}) \right\}$ $\qquad$ $\langle q|p\rangle = \exp\left\{ i \sum_a p_a q_a \right\}$

Euclidean QM / QFT

e^{-iHt} t: Minkowski or real time $\xrightarrow{\text{REPLACE WITH}}$ e^{-tH} t: Euclidean or imaginary time

Why?

(1) Numerical simulations: efficient algorithms exist only for Euclidean QFT

(2) Theoretical motivation: when time is discretized, renormalization and continuum limit are well understood only in the Euclidean

Doesn't this alter the physics?

(1) Hilbert space, Hamiltonian, vacuum, operators in the Schrödinger picture are the same as in the Minkowskian. The physics is essentially the same, it is just encoded in a different way in the time-dependent observables.

(2) Canonical ensamble: $\rho = \frac{1}{Z} e^{-\beta H}$ is the statical state of the system in thermodynamic equilibrium. Euclidean time = β = inverse temperature.
Euclidean QFT gives direct access to the thermodynamic properties of the system. [We will discuss this in detail.]

(3) Time-ordered Minkowskian and Euclidean n-point function are related by analytic continuation (Wick rotation).

Wick rotation and Euclidean time

[Theory of analytic semigroups]

Theorem – Let H be a self-adjoint operator, and bounded from below i.e. $H \geq E_0$ for some $E_0 \in \mathbb{R}$.

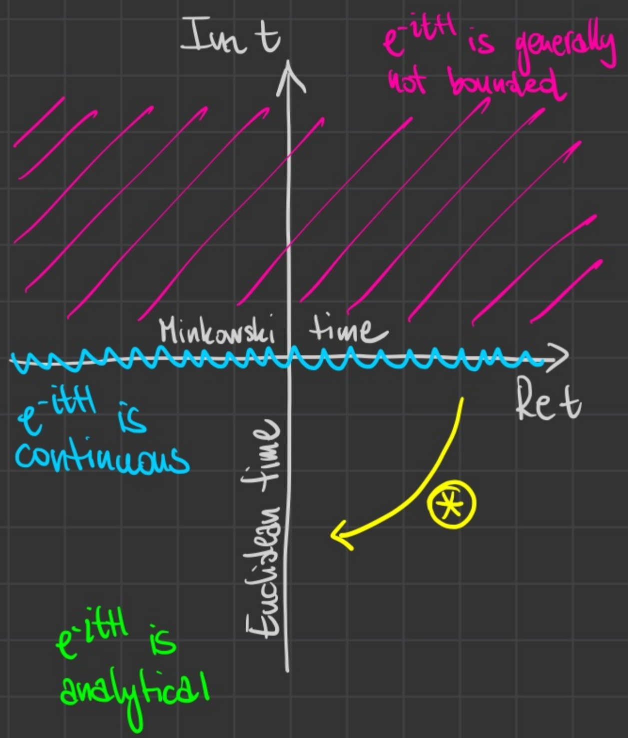
Then:

(1) The operator e^{-itH} is bounded, i.e. $\|e^{-itH}\| < +\infty$, for every complex t with $\operatorname{Im} t \leq 0$.

N.B. For $\tau > 0$, $t = i\tau$: $e^{-itH} = e^{\tau H}$ ✗ explodes with energy
$t = -i\tau$: $e^{-itH} = e^{-\tau H}$ ✓

In particular this implies that, for $\operatorname{Im} t \leq 0$

- e^{-itH} is defined on the whole Hilbert space $\mathcal{H}$
- e^{-itH} is a continuous operator



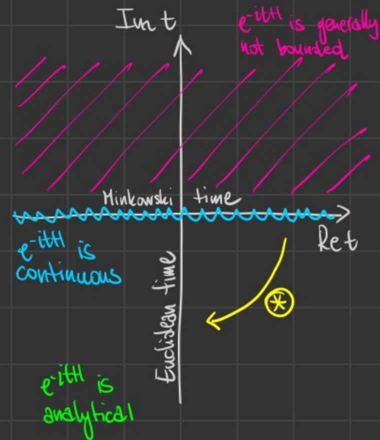
(2) The function $t \mapsto e^{-itH}$ is analytical for $\operatorname{Im} t < 0$.

In particular this implies that e^{-itH} can be expanded as a convergent power series around every t with $\operatorname{Im} t < 0$. The power series has a non-vanishing (generally finite) radius of convergence, and it converges in operator norm.

(3) The function $t \mapsto e^{-itH}$ is continuous (w.r.t. the operator norm) for $\operatorname{Im} t \leq 0$.

⊛ Wick rotation from positive Minkowski time to Euclidean time (via analytic continuation)

Example: time-ordered n-point functions (n=3 for simplicity)



$|\Omega\rangle$ = ground state of H = vacuum

Minkowskian 3-pt function $\quad C_M(x_1,x_2,x_3) = \langle\Omega|\,\hat{\varphi}_M(x_3)\,\hat{\varphi}_M(x_2)\,\hat{\varphi}_M(x_1)\,|\Omega\rangle$

$$= \langle\Omega|\, e^{ix_3^0 H}\,\hat{\varphi}(\underline{x}_3)\, e^{-i(x_3^0-x_2^0)H}\,\hat{\varphi}(\underline{x}_2)\, e^{-i(x_2^0-x_1^0)H}\,\hat{\varphi}(\underline{x}_1)\, e^{-ix_1^0 H}\,|\Omega\rangle$$

$$= e^{i(x_3^0-x_1^0)E_0}\,\langle\Omega|\,\hat{\varphi}(\underline{x}_3)\, e^{-i\underbrace{(x_3^0-x_2^0)}H}\,\hat{\varphi}(\underline{x}_2)\, e^{-i\underbrace{(x_2^0-x_1^0)}H}\,\hat{\varphi}(\underline{x}_1)\,|\Omega\rangle$$

$\varphi = 0$

Assume time-ordering: $x_3^0 > x_2^0 > x_1^0$ $\qquad$ positive time differences

Analytic continuation via Wick rotation: $\quad x_k^0 \xrightarrow{\text{REPLACE WITH}} e^{-i\varphi}\, x_k^0 \qquad \varphi \in [0, \pi/2]$

$\varphi = \pi/2$

Euclidean 3-pt function $\quad C_E(x_1,x_2,x_3) = \langle\Omega|\,\hat{\varphi}_E(x_3)\,\hat{\varphi}_E(x_2)\,\hat{\varphi}_E(x_1)\,|\Omega\rangle$

$$= e^{(x_3^0-x_1^0)E_0}\,\langle\Omega|\,\hat{\varphi}(\underline{x}_3)\, e^{-(x_3^0-x_2^0)H}\,\hat{\varphi}(\underline{x}_2)\, e^{-(x_2^0-x_1^0)H}\,\hat{\varphi}(\underline{x}_1)\,|\Omega\rangle$$